FINAL ASSIGNMENT
MODULAR FORMS 2019
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Instructions: You may not cooperate with anyone on the project

$\boxed{\text{Due Date: June 26,2019.}}$

Put the completed assignment in my mailbox (keep a copy).

Exercise 1. CANCELLED
~~Assume that there is a 12-dimensional even self-dual lattice. What would be its theta function? Show that there are no even self-dual lattices of dimension 12.~~

Exercise 2. Let $L' = \{(x, y) \in \mathbb{Z}^2 : x + y = 0 \bmod 5\} \subset \mathbb{Z}^2$. Find the Hermite normal form $L' = \langle(a, b),(0, d)\rangle$ for L', where $a, d \geq 1$, $ad = [\mathbb{Z}^2 : L']$ and $0 \leq b < d$.

Exercise 3. If $f \in S_k$ is a cuspidal Hecke eigenform, with Hecke eigenvalue $T(n)f = \lambda_f(n)f$, show that for prime p, and $|X| < p^{-k/2}$,

$$\sum_{j=0}^{\infty} \lambda_f(p^j)X^j = \frac{1}{1 - \lambda_f(p)X + p^{k-1}X^2}$$

Exercise 4. Let $\sigma_\alpha(n) = \sum_{d|n} d^\alpha$. We saw that it is a multiplicative function: $\sigma_\alpha(mn) = \sigma_\alpha(m)\sigma_\alpha(n)$, if m, n are coprime. Show that for p prime and $r \geq 1$,

$$\sigma_\alpha(p)\sigma_\alpha(p^r) = \sigma_\alpha(p^{r+1}) + p^\alpha\sigma_\alpha(p^{r-1})$$

Exercise 5. Suppose $f = A(0) + \sum_{m\geq1} A(m)q^m \in M_k$ is a non-cuspidal modular form of weight k (so $A(0) \neq 0$). We showed that $T(n)f = \sum_{m\geq0} B_n(m)q^m$ with

$$B_n(0) = A(0)\sigma_{k-1}(n)$$

and

$$B_n(m) = \sum_{d|\gcd(m,n)} d^{k-1}A(\frac{mn}{d^2}), \quad m \geq 1$$

a) Show that the Eisenstein series $E_k(\tau) = 1 + \gamma_k \sum_{n\geq1} \sigma_{k-1}(n)q^n$ is an eigefunction of all Hecke operators.

b) Show that if $f \in M_k \backslash S_k$ is a non-cuspidal eigenform of all Hecke operators $T(n)f = \lambda_f(n)f$, $n \geq 1$, then $\lambda_f(n) = \sigma_{k-1}(n)$ and $f = cE_k$.

Exercise 6. Denote by $D = \Delta/(2\pi)^{12} = q - 24q^2 + 252q^3 + \cdots = \sum_{n\geq 1} \tau(n)q^n$ the normalized modular discriminant, and $E_4 = 1 + 240\sum_{n\geq 1}\sigma_3(n)q^n$ the normalized Eisenstein series of weight 4.

a) Explain why the space S_{24} of cusp forms of weight 24 is spanned by

$$f_1 := DE_4^3 = q + 696q^2 + 162252q^3 + 12831808q^4 + \ldots$$

and

$$f_2 = D^2 = q^2 - 48q^3 + 1080q^4 + \ldots.$$

b) Find the coefficients c, d in the expansion $T(2)f_2 = cf_1 + df_2$.
Optional: Compute the matrix of the Hecke operator $T(2)$ in this basis.

c) Let $g_+, g_- \in S_{24}$ be two linearly independent eigenvectors of $T(2)$. Explain why $g_\pm$ are eigenvectors of all the Hecke operators $T(n)$.